Sensitivity Analysis of Discrete Structural Systems

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Nomenclature

A,B	= matrices defining general eigenvalue problems, Eq. (16)
a_{ik},b_{ik}	= coefficients in modal expansion of eigenvector derivatives, Eqs. (22-24)
F	= force vector, Eqs. (1) and (27)
f	= objective function
f^*	= optimum value of objective function
G	= stress-temperature matrix, Eq. (10)
g	= design constraint function, Eq. (2)
h	= finite difference step size
J	= Jacobian matrix
K	= stiffness matrix or Green's function
m	= total number of constraint functions in optimization problem, Eq. (38)
P	= parameter in optimization problem
p	= time-dependent design constraint, Eq. (29)
R_v	$= \frac{\partial F}{\partial v} - \frac{\partial K}{\partial v} U$
S	= stress-displacement matrix, Eq. (10)
T	= temperature
t	= time
t_f	= final time, Eq. (29)
U	= displacement vector, Eq. (1)
v	= design variable
v^*	= optimum value of design variable
X,Y	= eigenvectors or eigenvector matrices
λ	= adjoint variable vector, Eq. (6)
λ	= eigenvalue, Eq. (16)
λ_j	= Lagrange multiplier associated with jth constraint, Eq. (39)
σ	= stress, Eq. (10)
τ	= time

Introduction

SENSITIVITY analysis is emerging as a fruitful area of engineering research. The reason for this interest is the recognition of the variety of uses for sensitivity derivatives. In its early stages, sensitivity analysis found its predominant use in assessing the effect of varying parameters in mathematical models of control systems; see, for example, the texts of Tomovic,[1] Brayton and Spence,[2] Frank,[3] and Radanovic[4] for discussions of the early development of sensitivity theory. Interest in optimal control in the early 1960s (see, for example, Ref. 5) and automated structural optimization (see for example, Ref. 6) led to the use of gradient-based mathematical programming methods in which derivatives were used to find search directions toward optimum solutions. More recently, there has been strong interest in promoting systematic structural optimization as a useful tool for the practicing structural design engineer on large problems—a process still under way. Early attempts to use formal optimization for large structural systems resulted in excessively long and expensive computer runs. Examination of the optimization procedures indicated that the predominant contributor to the cost and time was the calculation of derivatives. As a consequence, emerging interest in sensitivity analysis has emphasized efficient computational procedures. In addition, researchers have developed and applied sensitivity analysis for approximate analysis, analytical model improvement, and assessment of design trends—so that structural sensitivity analysis has become more than a utility for optimization and is a versatile design tool in its own right. Most recently, researchers in disciplines such as physiology,[7] thermodynamics,[8] physical chemistry,[9] and aerodynamics,[10-12] have been using sensitivity methodology to assess the effects of parameter variations in their analytical models and to create designs insensitive to parameter variation.[13,14]

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This paper is a survey of methods applicable to the calculation of structural sensitivity derivatives for finite element modeled structures. Except for citing several general references, the paper does not deal with continuous (distributed parameter) models. The survey, which principally discusses literature published during the past two decades, concentrates on four main topics: derivatives of static response (displacements and stresses), eigenvalues and eigenvectors, transient response, and derivatives of optimum structural designs with respect to problem parameters. The bulk of the survey deals with derivatives of the aforementioned responses with respect to gage-type variables such as rod cross-sectional areas, beam cross-sectional dimensions, and plate thicknesses. Additionally, some works are reviewed in which the derivatives are calculated with respect to variables that define the shape of structural elements. Methods for calculating structural sensitivity derivatives are summarized in Table 1.

Sensitivity of Static Response

General Equations

This section of the paper focuses on the calculation of derivatives of static structural response (displacements and stresses) computed from finite element models. The governing equation for displacement is

$$KU=F \tag{1}$$

where K is the symmetric stiffness matrix of order $n \times n$, U the vector of displacements, and F the vector of applied forces. Both K and F are, in general, functions of design variables v. A typical function of displacement (e.g., a constraint) will be represented as

$$g=g(U,v), \qquad U=U(v) \tag{2}$$

Table 1 Summary of analytical methods[a] for structural sensitivity derivatives

Type of derivative	Method	Selected references
Static displacement		
WRT sizing variables	Direct	24, 23, 22
	Adjoint variable	25, 5, 23
WRT shape variables	Differentiate discrete equations	35, 36
	Material derivative	41, 42, 69, 55
Second derivatives	Direct	58, 57
	Adjoint variable	54, 55
	Mixed	56
Eigenvalues		
Symmetric matrices		
Distinct eigenvalues	Direct	80, 81, 72, 79
Mutiple eigenvalues	Direct	73, 74, 76
Nonsymmetric matrices	Direct	71, 83-85
Second derivatives	Direct	85-87, 94
Eigenvectors		
First derivatives	Direct	80, 83, 95, 68
	Modal expansion	80, 83, 84
Second derivatives	Direct	89
	Modal expansion	89
Transient displacement		
First derivatives	Direct	120
	Adjoint variable	115
	Green's function	112, 9
	FAST	121
Second derivatives	Direct	132
	Adjoint variable	133
Optimum designs		
Objective function		137, 143-145, 142
Design variables		147, 148, 146

[a]Finite difference methods are generally applicable; see, for example, Refs. 15-19.

Finite Difference Method

A straightforward method of calculating derivatives of g is to use a finite difference approximation. For example,

$$\frac{\mathrm{d}g}{\mathrm{d}v} \simeq \frac{g[U(v+h),v+h]-g[U(v),v]}{h} \tag{3}$$

A serious shortcoming of the finite difference method is the uncertainty in the choice of a perturbation step size h. If the step size is too large, truncation errors may be excessive. These can be thought of as errors due to retention of only the lowest-order terms of a Taylor series representation of a perturbed function. If the step size is too small, condition errors may occur. Condition errors are due to inaccuracies in the calculation of the displacements and roundoff errors in the finite difference calculation. Gill et al.[15,16] developed an algorithm to determine the optimum finite difference step size; i.e., one that balances the truncation and condition errors. The algorithm is based on approximating the truncation error as a linear function of step size h and the condition error as a linear function of $1/h$. This technique has been checked on functions that could be differentiated analytically and was found to be very effective. Other work on finding optimum step sizes was done by Stewart,[17] Kelley and Lefton,[18] and Haftka and Malkus.[19] A recent paper by Haftka[20] describes a technique for reducing condition errors in finite difference derivatives of response quantities obtained by iterative methods.

Analytical Methods

Analytical calculations of derivatives of displacements and their functions have been described by Arora and Haug[21,22] and Haug and Arora.[23] In these references, three methods are described: the direct or design space method,[24] the adjoint variable or state space method, and the virtual load method.[25] The virtual load method is a special case of the direct method. Both the direct and adjoint methods begin with the differentiation of Eqs. (1) and (2).

$$K\frac{\mathrm{d}U}{\mathrm{d}v}=\frac{\partial F}{\partial v}-\frac{\partial K}{\partial v}U \equiv R_v \tag{4}$$

$$\frac{\mathrm{d}g}{\mathrm{d}v}=\frac{\partial g}{\partial v}+\left(\frac{\partial g}{\partial U}\right)^T\frac{\mathrm{d}U}{\mathrm{d}v} \tag{5}$$

Direct Method

The direct method is to solve Eq. (4) for $\mathrm{d}U/\mathrm{d}v$ and substitute $\mathrm{d}U/\mathrm{d}v$ into Eq. (5). Equation (4) needs to be solved once for each design variable v so that the direct method is costly when the number of design variables is large.

Adjoint Method

The adjoint variable or state space method has been extensively used in optimal control theory; see, for example, Kelley.[5] The method starts by defining a vector of adjoint variables that satisfies the equation

$$K\lambda=\frac{\partial g}{\partial U} \tag{6}$$

where $\partial g/\partial U$ is sometimes referred to as the dummy load vector. (If g is a particular displacement component, then $\partial g/\partial U$ corresponds to a force of unit magnitude in the direction of the component.) Then using Eqs. (4-6),

$$\frac{\mathrm{d}g}{\mathrm{d}v}=\frac{\partial g}{\partial v}+\lambda^T R_v \tag{7}$$

The adjoint variable method requires the solution of Eq. (6) once for each function g. Therefore, if the number of functions is smaller than the number of design variables, the adjoint variable method is more efficient and, conversely, if the number of design variables is smaller, the direct approach is more efficient. Both the direct and adjoint methods involve fewer computations than the finite difference approach, which requires repeated factorization of the stiffness matrix, whereas the direct and adjoint methods require a single factorization with several right-hand sides.

Chon[26] developed a variant of the adjoint method via strain energy distribution and implemented it in a proprietary version of NASTRAN. Hsieh and Arora[27] and Gurdal and Haftka[28] extended the adjoint method for boundary conditions that require specialized treatment (such as design-variable-dependent boundary conditions), while Haug and Choi[29] suggest a generalization of the adjoint method that eliminates many of the problems associated with multipoint boundary conditions. Adaptation of the adjoint variable method to substructured finite element models is described by Arora and Govil.[30]

Calculation of $\partial K/\partial v$

An important computational task in the adjoint and direct methods is the calculation of $\partial K/\partial v$. If the structural model contains only elements whose stiffness matrix is proportional to v (such as rods where v is the cross-sectional area or membranes and shear panels where v is the thickness), $\partial K/\partial v$ is a constant matrix. But for elements having bending stiffness such as beams and plates, the stiffness matrix is a nonlinear function of the cross-sectional dimensions, and the stiffness matrix derivatives are not easily evaluated.[31] Hence, the preferred approach is to compute $\partial K/\partial v$ by finite differences as in Prasad and Emerson,[32] Camarda and Adelman,[33] and Wallerstein.[34]

Derivatives with Respect to Shape Design Variables

Shape design variables typically control the shape of the boundary of the structure—for example, variables controlling the shape of the hole (and thereby the stress concentration factor at the hole boundary). Differentiating the finite element equations to obtain Eq. (4) has two disadvantages. First, even small changes of the boundary can change the entire finite element mesh and, therefore, the calculation of $\partial K/\partial v$ is quite costly. Second, changes in shape can lead to the distortion of the finite elements and reduced accuracy. Thus, the derivatives obtained from Eq. (4) have a spurious component that reflects the changing accuracy of the solution when the mesh is distorted.[35,36] However, computational experience to date has failed to show that this spurious component presents serious problems.

There has been substantial work in obtaining sensitivity derivatives by differentiating the continuum equations, using the concept of material derivatives, and only then discretizing the problem. The reverse order of differentiation and discretization avoids spurious errors due to mesh distortion. Also, the derivatives can be expressed in terms of boundary integrals, which are cheaper to calculate than the term $\partial K/\partial v \cdot U$ because they are localized to the changing boundaries. Chun and Haug,[37-39] Rousselet and Haug,[40,41] Rousselet,[42] Zolesio,[43] Choi and Haug,[44] Dems and Mróz,[45] Yoo, Haug, and Choi,[46] and Choi[47] have proposed formulations using boundary integrals and the adjoint method. Unfortunately, there are considerable numerical difficulties associated with the evaluation of boundary integrals,[48] especially for low-order elements that do not model a curved boundary well.

Braibant and Fleury[49] avoided these numerical difficulties by using domain instead of boundary integrals in a direct approach formulation. Such a domain integral approach was also used with the adjoint formulation by Choi and Haug,[50] Hou et al.,[51] and Choi and Seong.[52] The domain integrals avoid the numerical difficulties associated with boundary integrals, but they are about as expensive as the use of Eq. (4).

The main advantage of the continuum approach seems to be the generality of its results. It is equally applicable to finite element, boundary element, or any other numerical or analytical solution technique.

Calculation of Second Derivatives

Second derivatives of displacement and constraint functions are used for approximate analysis,[53] and for the calculation of derivatives of optimal solutions (see subsequent section on this topic). Such derivatives may be obtained by differentiating Eqs. (4) and (5), for example,

$$K\frac{d^2U}{dv^2}=\frac{\partial R_v}{\partial v}+\frac{\partial R_v}{\partial U}\frac{dU}{dv}$$

$$\frac{d^2g}{dv^2}=\frac{\partial^2 g}{\partial v}+2\left(\frac{\partial^2 g}{\partial U\partial v}\right)^T\frac{dU}{dv}+\left(\frac{\partial g}{\partial U}\right)^T\frac{d^2U}{dv^2} \tag{8}$$

However, for m design variables there are $m(m+1)/2$ second derivatives, and Eqs. (8) need to be solved for that many right-hand sides. It is possible to proceed with an extension of the adjoint variable method proposed by Haug,[54] and Dems and Mróz.[55] However, a more efficient approach proposed by Haftka[56] is to use Eq. (6) to obtain

$$\frac{d^2g}{dv^2}=\frac{\partial^2 g}{\partial v^2}+2\left(\frac{\partial^2 g}{\partial U\partial v}\right)\frac{dU}{dv}+\lambda^T\left(\frac{\partial R_v}{\partial v}+\frac{\partial R_v}{\partial U}\frac{dU}{dv}\right) \tag{9}$$

This approach requires the solution of Eq. (4) for all the first derivatives and Eq. (6) for all vectors of adjoint variables.

Second derivatives were also derived by Van Belle,[57] using flexibility rather than stiffness matrices. Finally, Jawed and Morris[58] described a procedure for approximating higher-order derivatives from the first derivative information, which is equivalent to introducing intermediate variables.

Stress Derivatives

The stresses in an element may be obtained from the displacements using

$$\sigma=SU-GT \tag{10}$$

where σ is a vector of element stresses, T is an element temperature, and S and G are stress-displacement and stress-temperature matrices, respectively.

Derivatives of stresses may be obtained by differentiating Eq. (10)

$$\frac{d\sigma}{dv}=S\frac{dU}{dv}+\frac{\partial S}{\partial v}U-\frac{\partial G}{\partial v}T \tag{11}$$

For finite elements such as rods, membranes, and shear panels, S and G are independent of v, and stress derivatives are obtained by simply substituting dU/dv into Eq. (11). For bending-type elements, S and G may be functions of v and the complete expression must be used; see Camarda and Adelman.[33]

Nonlinear Analysis

When geometric or material nonlinearities are important, Eq. (1) is no longer valid, and the displacement U is calculated from a system of the form

$$F(U,v)=0 \tag{12}$$

where F is a vector of nonlinear functions. Derivatives are obtained by differentiating Eq. (12) with respect to v

$$J\frac{dU}{dv}=-\frac{\partial F}{\partial v}\equiv R_v \tag{13}$$

where the Jacobian J is $\partial F/\partial U$ (often referred to as the tangential stiffness matrix). The derivative of any constraint g may be calculated by solving Eq. (13) for dU/dv and then substituting into Eq. (5)—this is the direct method. Alternatively one can solve for the adjoint vector λ from

$$J^T\lambda = \frac{\partial g}{\partial U} \tag{14}$$

and calculate dg/dv from Eq. (7) using R_v from Eq. (13).

Applications

Applications of displacement sensitivity derivatives for formal optimization are described, for example, in Nguyen and Arora,[59] Arora,[60] Prasad and Haftka,[61] and Schmit and Farshi.[62] Use of displacement and stress derivatives to construct explicit constraint approximations is described, for example, by Schmit and Farshi,[62] Storaasli and Sobieszczanski,[63] and Noor and Lowder.[53] A basic example of such an approximation is

$$U(v^*) \simeq U(v) + \frac{dU}{dv}\Delta v \tag{15}$$

where $U(v)$ is the displacement vector for the design variable v, $U(v^*)$ is the vector corresponding to the new design variable $v^* = v + \Delta v$. Numerous examples of application of stress derivatives in formal optimization are cited in the survey by Schmit.[6] Less well known is the use of sensitivity derivatives of stresses to effect design changes without formal optimization. A good example of this is reported by Musgrove et al.[64] The most common applications of sensitivity calculations in nonlinear static response are of derivatives of U with respect to a load parameter. Such derivatives are useful in incremental solution procedures of Eq. (12) or for reduced basis solution of this equation.[65] Finally, readers interested in the topic of static response sensitivity of distributed parameter systems are referred to Haug and Komkov,[66] Haug and Rousselet,[67] Haug,[68] Rousselet,[69] and Hsieh and Arora,[27] as well as the text of Haug, Komkov, and Choi.[70]

Sensitivity of Eigenvalues and Eigenvectors

The general problem is to compute derivatives of eigenvalues and eigenvectors with respect to design variables or system parameters. For reference purposes, the most general case considered is the following eigenvalue problem:

$$AX = \lambda BX \tag{16}$$

$$Y^TA = \lambda Y^TB \tag{17}$$

$$Y^TBX = 1 \tag{18}$$

where λ is an eigenvalue (generally complex). The generally nonsymmetric real $n \times n$ matrices A and B are assumed to be explicit functions of a set of design variables v. And X and Y are right and left eigenvectors, respectively. The first result on eigenvalue derivatives was published by Jacobi,[71] who developed the result for the special case of symmetric A, and $B = I$

$$\frac{\partial\lambda}{\partial v} = Y^T\frac{\partial A}{\partial v}X \tag{19}$$

Wittrick[72] applied Jacobi's formula for the case of a symmetric matrix to the derivatives of buckling eigenvalues and presented results for the change in buckling loads of plates with respect to aspect ratio and thickness. Lancaster[73] developed a rigorous treatment of eigenvalue derivatives and, in particular, showed that for multiple eigenvalues the derivatives themselves are solutions of an eigenvalue problem. The issue of multiple eigenvalues was also investigated by Simpson,[74] Bratus and Seyranian,[75] and Haug and Rousselet,[76] who showed that while simple eigenvalues are differentiable (Frechet), multiple eigenvalues are only directionally (Gateaux) differentiable.

Two methods developed for sensitivity analysis of electronic networks are notable for their nonreliance on eigenvectors in the eigenvalue derivative formulas. Rosenbrock[77] and Morgan[78] developed formulas for eigenvalue derivatives in terms of the matrix A and its eigenvalues. According to Morgan's own assertion, however, the computational effort is not much less than if eigenvectors were required, and examination of the details of their methods indicates that the calculations are equivalent to those required for computing eigenvectors.

Other contributions from the electronics discipline include the use of the adjoint network theory. An adjoint network or structure is one with the same geometry and nodal connections as the actual configuration, but the elements of the adjoint system may be linear even though the actual elements are nonlinear. Vanhonacker[79] has used the theory of adjoint structures to derive formulas for derivatives of eigenvalues and eigenvectors of structures.

Fox and Kapoor[80] and Fox[81] considered the special case of symmetric A and B matrices but developed techniques applicable to more general cases. For eigenvalues their formula is

$$\frac{\partial\lambda}{\partial v} = X^T\left(\frac{\partial A}{\partial v} - \lambda\frac{\partial B}{\partial v}\right)X \tag{20}$$

in which it is assumed that the eigenvectors are normalized such that

$$X^TBX = 1 \tag{21}$$

For eigenvector derivatives, two methods are presented by Fox and Kapoor. The first is to differentiate Eq. (16), giving a set of simultaneous equations for the eigenvalue and eigenvector derivatives. A complication here is that the equations for the eigenvector derivatives are singular and the set is solvable only after algebraic manipulation, which destoys the banded nature of the equations, a point that arises later in connection with another method. The second method for eigenvector derivatives, developed by Fox and Kapoor, is to expand the derivative as a series of eigenvectors. Thus, for the ith eigenvector

$$\frac{\partial X_i}{\partial v} = \sum_{k=1}^{n} a_{ik}X_k \tag{22}$$

The coefficients a_{ik} are obtained by substituting Eq. (22) into equations resulting from differentiating Eq. (16). In principle, it is necessary to use all n modes in the expansion of Eq. (22). However, as with the modal method generally, it should be possible to obtain reasonable results with fewer than n eigenvectors. Study of the convergence properties of Eq. (22) is clearly called for. Fox and Kapoor's second method was specialized by Hirai and Kashiwaki[82] for the case of design variables controlling only a small part of the structure. Rogers[83] and Stewart[84] derived sensitivity formulas for eigenvalues and eigenvectors of the general problem [Eqs. (16) and (17)]. For eigenvalues the equation is

$$\frac{\partial\lambda}{\partial v} = Y^T\left(\frac{\partial A}{\partial v} - \lambda\frac{\partial B}{\partial v}\right)X \tag{23}$$

Rogers expressed the eigenvector derivatives as an expansion in terms of the eigenvectors

$$\frac{\partial X_i}{\partial v} = \sum_{k=1}^{n} a_{ik}X_k\,, \qquad \frac{\partial Y_i}{\partial v} = \sum_{k=1}^{n} b_{ik}Y_k \tag{24}$$

The coefficients a_{ik} and b_{ik} are computed by substituting Eqs. (24) into an expression obtained by differentiating the eigenvalue problem and combining it with appropriate orthogonality conditions. Plaut and Husseyin,[85] as well as Rudisill[86] and Doughty,[87] developed the same results as Rogers and, in addition, developed a formula for second derivatives of eigenvalues. Miura and Schmit[88] developed an approximate expression for second derivatives of real eigenvalues suitable for numerical optimization procedures. Formulas for the second derivatives of eigenvectors are presented by Taylor and Kane.[89] Garg[90] investigated the case where A and B were complex and produced formulas for the eigenvalue and eigenvector derivatives. Garg's eigenvector derivative procedures are analogous to those of Fox and Kapoor. Rudisill and Chu[91] developed the same eigenvalue derivative formulas as Rogers. Additionally, for eigenvector derivatives they extended Fox and Kapoor's first formulation to the case where A and B are nonsymmetric. They suggest two ways to solve the equations for the derivatives: an iterative method that converges to the derivatives of the lowest eigenvalue and corresponding eigenvector; and an algebraic method that is an extension of Fox and Kapoor's first method. Andrew[92,93] provided some proofs and refinements of Rudisill's and Chu's algorithm. Brandon[94] showed that the second derivatives of eigenvalues may be calculated by using the first derivatives of the eigenvectors.

An alternate method for calculation of eigenvector derivatives is due to Nelson.[95] Differentiating the eigenvalue problem of Eq. (16) gives

$$(A-\lambda B)\frac{\partial X}{\partial v}=-\left(\frac{\partial A}{\partial v}-\frac{\partial \lambda}{\partial v}B-\lambda\frac{\partial B}{\partial v}\right)X \tag{25}$$

The matrix $A-\lambda B$ is singular since λ is an eigenvalue. The method of Nelson is to represent the eigenvector derivative as

$$\frac{\partial X}{\partial v}=V+cX \tag{26}$$

where V is the solution of a reduced version of Eq. (25) obtained by deleting the kth row and column from $A-\lambda B$ (where k is chosen arbitrarily) and setting the kth component of V equal to zero. The multiplier c is evaluated by substituting Eq. (26) into an equation obtained by differentiating Eq. (21). This method has certain advantages over previous eigenvector derivative techniques: it requires only the eigenvalue and eigenvector for the mode being differentiated, and the equation for V retains the banded character of coefficient matrix (unlike the algebraic methods of Fox and Kapoor, Plaut and Husseyin, and Rudisill). Nelson's method was put in a more general setting in terms of the generalized inverse of $A-\lambda B$ by Chen and Wei.[96] Cardani and Mantegazza[97] extended Nelson's method to transcendental flutter eigenvalue problems. Flutter eigenvalue derivatives were also derived by Rudisill and Bhatia,[98] Rao,[99] Seyranian,[100] and Pedersen and Seyranian.[101] Derivatives of nonlinear buckling eigenvalues were obtained by Kamat and Ruangsiliasingha.[102] Finally, for the sensitivity analysis of eigenvectors of distributed parameter systems, papers by Farshad[103] and Haug and Rousselet[76] and the text by Haug, Komkov, and Choi[70] should be of interest to readers.

Sensitivity of Transient Response

General

The discussion of sensitivity analysis of transient structural response is usually based on the equations of motion written as a system of second-order differential equations. However, this form obscures the similarity of structural sensitivity analysis to sensitivity analysis in other fields where first-order differential equations are employed and is also less compact than a first-order formulation. For these reasons the discussion will be based on a system of first-order ordinary differential equations of the form

$$\dot{U}=F(U,t,v)$$
$$U(0)=U_0 \tag{27}$$

where U is the response, F a vector of functions, t time, and v a typical design parameter; a dot denotes differentiation with respect to time. In many structural applications the left-hand side of Eqs. (27) is $A\dot{U}$, where A is a matrix, and the methods discussed below are also applicable to that more general form; see, for example, Ref. 104.

Direct Method

The direct method of obtaining sensitivity derivatives is based on differentiating Eqs. (27) to obtain

$$\frac{\mathrm{d}\dot{U}}{\mathrm{d}v}-J\frac{\mathrm{d}U}{\mathrm{d}v}=\frac{\partial F}{\partial v}$$
$$\frac{\mathrm{d}U}{\mathrm{d}v}(0)=0 \tag{28}$$

where the Jacobian J is $\partial F/\partial U$. Note that Eqs. (28) are a system of linear differential equations, even if the original system, Eqs. (27), is nonlinear. Often derivatives of the entire vector U are not required. Instead it is necessary to obtain the derivatives of a function of U of the form

$$g(U,v)=\int_0^{t_f}p(U,t,v)\mathrm{d}t \tag{29}$$

where p is a functional representation of a time-dependent constraint and t_f is a final time for the response calculation. The direct approach obtains $\mathrm{d}g/\mathrm{d}v$ as

$$\frac{\mathrm{d}g}{\mathrm{d}v}=\int_0^{t_f}\left[\frac{\partial p}{\partial v}+\left(\frac{\partial p}{\partial U}\right)^T\frac{\mathrm{d}U}{\mathrm{d}v}\right]\mathrm{d}t \tag{30}$$

where $\mathrm{d}U/\mathrm{d}v$ is calculated in Eqs. (28).

Green's Function Method

Equations (28) have to be solved once for each design variable and are costly when the number of design variables is large. When the number of design variables is larger than the dimensionality of U, then the Green's function approach[9] is more efficient than the direct approach. An application of this approach is sensitivity analysis of transient structural response, when the response is computed using reduction techniques such as modal analysis (e.g., Refs. 104 and 105). The sensitivity derivative $\mathrm{d}U/\mathrm{d}v$ is written as

$$\frac{\mathrm{d}U}{\mathrm{d}v}(t)=\int_0^{t}K(t,\tau)\frac{\partial F}{\partial v}(\tau)\mathrm{d}\tau \tag{31}$$

where the Green's function K satisfies (recall that the dot denotes $\mathrm{d}/\mathrm{d}t$)

$$K(t,\tau)=0, \qquad t<\tau$$
$$K(\tau,\tau)=I$$
$$\dot{K}(t,\tau)-J(t)K(t,\tau)=0, \qquad t>\tau \tag{32}$$

The efficiency of the Green's function approach is partly governed by the method used to integrate Eqs. (32). A large amount of work on the efficient implementation of the Green's function approach has been performed by Rabitz and co-workers.[106-113] Their approach is implemented in a general-purpose computer code called AIM.[112] The Green's function method is also known as the variational method.[114]

Adjoint Variable Method

Further improvements in efficiency may be possible if less information is needed. If instead of the derivatives of the entire vector U, only those of a few functionals [e.g., Eq. (29)] are required, then an adjoint variable method is called for. The adjoint variable approach solves first for the adjoint vector λ from the differential equation

$$\dot{\lambda}+J^T\lambda=\frac{\partial p}{\partial U}$$

$$\lambda(t_f)=0 \tag{33}$$

It is shown by Haftka and Kamat[104] that

$$\frac{\mathrm{d}g}{\mathrm{d}v}=\int_0^{t_f}\left(\frac{\partial p}{\partial v}-\lambda^T\frac{\partial F}{\partial v}\right)\mathrm{d}t \tag{34}$$

Equations (33) are a set of linear differential equations that is integrated backward from t_f to zero. As in the steady-state case, the adjoint variable approach is preferred over the direct approach when the number of functionals is less than the number of design variables. The adjoint variable approach has been applied to a variety of problems, including dynamics,[115-117] atmospheric diffusion,[118] nuclear processes,[119] and heat transfer in structures.[120]

Finite Difference Method

For sensitivity analysis of static response, the finite difference approach is almost always inferior to analytical methods. For the calculation of derivatives of transient response, this is not always the case. When explicit methods are used for integrating the differential equations, the linearity of the sensitivity equations does not constitute a computational advantage. Therefore, for the case of explicit integration, the finite difference approach is often computationally superior to the direct method (see Refs. 120 and 19). When implicit integration techniques are used, the finite difference approach is less attractive computationally but remains easier to implement than the direct approach.

FAST Method

All the approaches discussed above provide local sensitivity information. The Fourier amplitude sensitivity test (FAST) method,[121] which provides global sensitivities, is typically used to assess sensitivities to parameter uncertainties. By systematically sampling solutions obtained by varying the parameters that have a range of uncertainty. If there are m parameters v_i, $i=1,\ldots,m$, the sampling is performed in an m-dimensional space. FAST converts this m-dimensional space to a one-dimensional space in terms of a variable s by using the transformation

$$v_i=a_i+b_i\sin w_i s \tag{35}$$

where w_i, $i=1,\ldots,m$ are a set of incommensurate frequencies and a_i, b_i are constants that depend on the range of variation v_i. The solutions for a large number of s values are sampled, and a Fourier transform of the response in terms of s is obtained. The coefficient of the transform associated with w_i is a direct measure of the sensitivity of the solution to v_i. While FAST is more efficient than a Monte Carlo sampling of the parameter space, it is substantially more expensive than local sensitivity methods when m is large.

While in the literature reviewed here, FAST has been used only for calculation of sensitivities of transient response, the method is equally applicable to steady-state or eigenproblem sensitivity calculations. The method has been applied extensively in physical chemistry,[122,123] and a computer implementation is described by McRae, Tilden, and Seinfeld.[124]

Other Forms of Transient Response Equations

A specialized form of transient structural response is the response to harmonic excitation. The sensitivity analysis of that response is very similar to the sensitivity analysis of static response (see, for example, Refs. 125-127).

The system of Eqs. (27) is typically obtained by discretization of the spatial variation (e.g., by finite elements) before the sensitivity analysis is performed. In some applications (see, for example, the discussion of static shape sensitivity), it may be advantageous to perform the sensitivity analysis before discretizing. Koda, Dogru, and Seinfeld,[128] Dwyer and Peterson,[10] and Koda and Seinfeld,[129] for example, discuss applications of sensitivity techniques to partial differential equations, while Gibson and Clark[130] and Cacuci[131] present sensitivity analysis in the general setting of functional analysis.

Second Derivatives

Part of the motivation for second derivatives is that they estimate nonlinear sensitivity effects including interaction between variables. Second derivatives may be calculated directly. For example, differentiating Eqs. (28),

$$\frac{\mathrm{d}^2\dot{U}}{\mathrm{d}v^2}-J\frac{\mathrm{d}^2U}{\mathrm{d}v^2}=\frac{\partial^2F}{\partial v^2}+2\frac{\partial J}{\partial v}\frac{\mathrm{d}U}{\mathrm{d}v} \tag{36}$$

Unfortunately m design parameters result in $m(m+1)/2$ systems such as Eq. (36). If second derivatives are needed only for a functional g such as Eq. (29), then the calculation can be greatly simplified. In fact,

$$\frac{\mathrm{d}^2g}{\mathrm{d}v^2}=\int_0^{t_f}\left(\left(\frac{\mathrm{d}U}{\mathrm{d}v}\right)^T\frac{\mathrm{d}^2p}{\mathrm{d}U^2}\frac{\mathrm{d}U}{\mathrm{d}v}-\Lambda^T\frac{\partial^2F}{\partial v^2}+2\frac{\partial J}{\partial v}\frac{\mathrm{d}U}{\mathrm{d}v}\right)\mathrm{d}t \tag{37}$$

Thus, the solution for all the second derivatives requires only first derivatives of U plus the adjoint variable vector. This efficient approach to second-order sensitivity calculations is not yet in use. The literature describes somewhat less efficient direct and adjoint techniques[132,133] or finite difference techniques.[134]

Sensitivity Derivatives of Optimal Solutions

As the use of optimization techniques has expanded, there has been increasing interest in the sensitivity of optimal solutions to problem parameters. A typical situation in which such derivatives are needed is the following: Suppose the minimum weight design of an aircraft wing is obtained by varying the sizes of the structural components while the geometry of the wing, the loading, and the structural materials are fixed during the optimization process. Now, suppose the minimum weight design is still too heavy and the designer needs to know which of the fixed parameters is a good candidate for change. It would be useful to have the sensitivity of the minimum weight design to changes in such parameters.

The information required for obtaining the sensitivity of an objective function such as minimum weight with respect to problem parameters is composed of a direct effect on the objective function plus an indirect effect through the change in the constraints. For example, if the optimization problem is posed as:

$$\text{Minimize}\, f(v)$$

such that

$$g_j(v)>0, \qquad j=1,\ldots,m \tag{38}$$

where $f(v)$ is an objective function, v is a vector of design variables, and $g_j(v)$ represent constraints. Let v^*, f^* be the solution to the problem and let P be a problem parameter. Then it is shown[135] that

$$\frac{df^*}{dP}=\frac{\partial f}{\partial P}(v^*)-\sum_{j=1}^{m}\lambda_j\frac{\partial g_j}{\partial P}(v^*) \tag{39}$$

where λ_j are the Lagrange multipliers associated with the constraints. The Lagrange multipliers thus have the role of the "price" of the constraints in that λ_j is the change in the objective function due to a unit change in g_j. Because most optimization algorithms yield the Lagrange multipliers or estimates thereof as a by-product of the solution, the sensitivity of the objective function to problem parameters is easy to obtain.

The sensitivity of the optimum set of design variables v^* with respect to problem parameters is more complicated. Lagrange multipliers are not sufficient, and additional calculations are required. Early work by Fiacco and McCormick,[136] Armacost and Fiacco,[137] Fiacco,[138,139] Bigelow and Shapiro,[140] and Robinson[141] concentrated on the mathematical aspects (see also text by Fiacco, Ref. 142). More recent papers by McKeown,[143,144] Sobieszczanski-Sobieski, Barthelemy, and Riley,[145] and Vanderplaats and Yoshida[146] discuss applications to the optimal design of dynamic systems and to structures. The calculation of the derivatives of v^* requires second derivatives of the objective function and constraints with respect to the design variables and thus poses a need for efficient computational techniques to obtain these derivatives.

As with other sensitivity derivatives, derivatives of optimal solution may be used to extrapolate solutions for problem parameter changes. Unfortunately, the sensitivity derivatives do not take into account changes in the active constraint set brought about by the change of parameters.[147] Consider, for example, a constraint that is almost but not quite critical for the optimum design. The Lagrange multiplier associated with the constraint must be zero and, therefore, as indicated in Eq. (34), such a constraint does not contribute to the sensitivity of the objective function. However, a small change in the value of P can make the constraint critical and completely change the value of the derivative. This problem makes the use of optimal solution sensitivity derivatives riskier than some other sensitivities. Sobieszczanski-Sobieski, Barthelemy, and Riley[145] suggested using derivatives of the Lagrange multipliers and the optimum solution vector v^* to anticipate changes in the active set. However, the effectiveness of this approach is still in doubt with positive results obtained by Schmit and Chang[148] and negative results by Barthelemy and Sobieski.[147]

Concluding Remarks

This article surveys methods for calculating sensitivity derivatives for discrete structural systems and primarily covers literature published during the past two decades. Methods are described for calculating derivatives of static displacements and stresses, eigenvalues and eigenvectors, transient structural response, and derivatives of optimum structural designs with respect to problem parameters. Methods and selected references are summarized in Table 1. The survey is focused on publications addressed to structural analysts but also includes a number of methods developed in nonstructural fields such as controls and physical chemistry, which are directly applicable to structural formulations. Most notable among the nonstructural-based methods are the adjoint variable techniques from control theory and the Green's function and FAST methods from physical chemistry.

For static displacements and stresses, methods are well established for derivatives with respect to simple sizing variables. Finite difference and analytical methods (direct and adjoint variable) are available, and there are clear guidelines giving classes of problems in which the various methods are preferred. Finite differences have long been a disparaged method as compared to the more elegant analytical approaches—and indeed the theoretical effort (as measured by operation counts, for example) of finite differences does greatly exceed that of the analytical approaches except for very small numbers of design variables. However, finite differences have a major advantage—they are extremely simple to formulate and implement. This factor, together with the increased speed of recent and expected computers, may explain the popularity of finite differences in many applications.

Methods for derivatives with respect to shape design variables are less well established, and consequently there are no clear choices of preferred techniques. One approach is to differentiate a set of discretized equations from a finite element model with respect to the shape design variables. Although this method has the advantage of versatility, it has this disadvantage: when the shape changes, the finite element mesh may be distorted, leading to numerical inaccuracies. An alternative approach is to differentiate the continuum equations (before discretization) using a material derivative. This approach avoids the mesh distortion problem and is potentially more efficient if more complex to implement.

With regard to derivatives of structural eigenvalue problems, well-established formulas are available for both real and complex eigenvalues. Derivatives of eigenvectors may be obtained by several methods including expanding the derivatives as a series of eigenvectors, an algebraic approach based on simultaneous equations for eigenvalue and eigenvector derivatives, and a simplified but rigorous analytical approach developed by Nelson. The method of Nelson is most appealing as it combines mathematical rigor with computational simplicity. The modal expansion method also merits consideration but requires a study of the convergence properties of the technique.

Derivatives of transient structural response may be obtained using finite differences, direct and adjoint variable analytical methods, a Green's function technique, and the Fourier amplitude sensitivity test (FAST); the latter two methods were developed by physical chemistry researchers. As in the static case, there are established guidelines for deciding when to choose among the various methods. Unlike the static case, the finite difference method may be competitive on the basis of computational efficiency. For example, if an explicit numerical integration algorithm is used for the nominal solution, a finite difference calculation of the derivative may be more efficient than an analytical method.

Methods for derivatives of optimum designs with respect to problem parameters are reviewed. Because this is a relatively new topic, the body of literature is not large. The derivative of the objective function can be easily obtained by a reasonably simple formula. The derivatives of the optimum design variables are somewhat more difficult to obtain. A complication that arises in using these derivatives to extrapolate an optimum design is that one must keep track of constraints that change from noncritical to critical as a result of small parameter changes. Finally, a significant by-product of the interest in derivatives of optimum designs is the motivation it has provided for research in improved methods for second derivatives of response quantities.

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